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Non-commutative multisolitons: moduli spaces, quantization, finite θ effects and stability

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Abstract: We find the N-soliton solution at infinite θ, as well as the metric on the moduli space corresponding to spatial displacements of the solitons. We use a perturbative expansion to incorporate the leading θ^{-1} corrections, and find an effective short range attraction between solitons. We study the stability of various solutions. We discuss the finite θ corrections to scattering, and find metastable orbits. Upon quantization of the two-soliton moduli space, for any finite θ, we find an s-wave bound state.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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1. Introduction

Recently [1] it was realized that one can construct stable soliton solutions in non-commutative scalar field theory even though such solitons do not exist in commutative scalar theories in higher than two dimensions. The solutions are particularly simple when the noncommutativity parameter $\theta \to \infty$, where one finds an infinite-dimensional moduli space of solitons. This program has also been extended to non-commutative gauge theories [2]–[6].

These solitons have found an application in the context of tachyon condensation where D-branes can be found as soliton solutions on higher-dimensional non-BPS D-branes. By turning on a B-field one makes these non-BPS D-branes non commutative and the soliton configurations studied represent various types of lower-dimensional D-branes [7, 8].

There also seems to be a place for application of these solitons in a non-commuta-tive description of the Quantum Hall Effect [9]–[15].

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Motivated by these developements, we have studied what happens when one scatters non-commutative solitons. In [16] we analyzed this questions using moduli space techniques, and found a Kähler metric on the moduli space somewhat analogous to the metric on the moduli space of two magnetic monopoles. A natural generalization of the results in [16] is to find the moduli space metric for N solitons. In this paper we find a simple and elegant expression for the Kähler potential for the general case.

The analysis in [1] was mainly done at infinite θ but a program to find corrections to the solitons at finite θ was initiated. This was followed by studies at finite θ, both numerically [17, 18] and theoretically [19, 20]. This topic is important, since one would like to know if the solitons are stable at finite θ, and if they are, how many of the infinite number of moduli survive. We study this issue and find that at finite θ, non-radial excitations, which were ignored in [17]–[20], destabilize all "excited" soliton states, and leave only the basic N-multisoliton solutions.

Some quantum issues have also been studied in [21, 22]. However, the discussion in [21] involves averaging over non-radial modes, which we find play an essential role.

This paper is organized as follows: in section 2, we construct the general N-soliton solution at infinite θ. We find the metric on the moduli space corresponding to spatial displacements of the solitons, and discuss the three-soliton case in detail. In section 3, we introduce a perturbative expansion that allows us to incorporate the leading θ^{-1} corrections. In the two-soliton case we find an effective short range attraction between solitons. In section 4 we use these perturbative results to study the stability of various solutions. In section 5 we focus on the two-soliton case and discuss the finite θ corrections to scattering. We find a range of interesting phenomena, including metastable orbits. In section 6, we quantize on the two-soliton moduli space, and find an s-wave bound state for any finite θ.

While writing of this paper, we became aware that results which have some overlap with our results were presented in [23].

2. Multisolitons at infinite θ

2.1 Multisoliton solutions

The two-soliton solution at infinite θ, constructed in [1], is

$$\Phi_2 = \lambda \left(|z_+\rangle \langle z_+| + |z_-\rangle \langle z_-| \right), \tag{2.1}$$

where λ is an extremum of the potential $V(\phi)$,

$$|z_\pm\rangle = \frac{|z\rangle \pm |-z\rangle}{\sqrt{2\left(1 \pm \mathrm{e}^{-2|z|^2}\right)}}, \tag{2.2}$$

and $|\pm z\rangle = \mathrm{e}^{-\frac{1}{2}|z|^2 \pm z a^\dagger}\,|0\rangle$. This can be generalized to the n soliton case as follows: let $z_\alpha, \alpha = 1, \ldots, n$ be pairwise different complex numbers satisfying the center of mass condition $\sum_{\alpha=1}^n z_\alpha = 0$. With

$$|z_\alpha\rangle \equiv \mathrm{e}^{-\frac{1}{2}|z_\alpha|^2 + a^\dagger z_\alpha}\,|0\rangle\,, \tag{2.3}$$

the multi-soliton solution is

$$\Phi_n = \lambda \sum_{\alpha,\beta=1}^{n} |z_\alpha\rangle\, A^{-1}_{\alpha\beta}\,\langle z_\beta| \equiv \lambda P_n\,, \tag{2.4}$$

where A is the $n \times n$ matrix,

$$A_{\alpha\beta} = \langle z_\alpha \mid z_\beta\rangle = e^{-\frac{1}{2}(|z_\alpha - z_\beta|^2 + z_\alpha \bar{z}_\beta - z_\beta \bar{z}_\alpha)}\,, \tag{2.5}$$

and P_n is a rank n projection operator onto the linear subspace of the harmonic oscillator Hilbert space $\mathcal{H}$ spanned by the vectors $|z_\alpha\rangle$.

For large separations, i.e. $|z_\alpha - z_\beta| \gg 1$ for all $\alpha \neq \beta$,

$$\Phi_n = \sum_{\alpha=1}^{n} \lambda\ |z_\alpha\rangle\,\langle z_\alpha| + \mathcal{O}\left(\mathrm{e}^{-\left|z_\alpha - z_\beta\right|^2}\right), \tag{2.6}$$

i.e. in this limit, Φ_n describes n well separated level 0 solitons. To study the limit $z_\alpha \to 0$ it is convenient to introduce a new basis:

$$\begin{aligned}
|u_1\rangle &= \mathrm{e}^{\frac{1}{2}\epsilon^2|z_1|^2}\,|\epsilon z_1\rangle = |0\rangle + \mathcal{O}(\epsilon)\,,\\
|u_2\rangle &= \frac{1}{\epsilon(z_2 - z_1)}\left(\mathrm{e}^{\frac{1}{2}\epsilon^2|z_2|^2}\,|\epsilon z_2\rangle - |u_1\rangle\right) = |1\rangle + \mathcal{O}(\epsilon)\,,\\
|u_3\rangle &= \frac{\sqrt{2}}{\epsilon^2(z_3 - z_1)(z_3 - z_2)}\left(\mathrm{e}^{\frac{1}{2}\epsilon^2|z_3|^2}\,|\epsilon z_3\rangle - |u_1\rangle - \epsilon(z_3 - z_1)\,|u_2\rangle\right) = |2\rangle + \mathcal{O}(\epsilon)\,,\\
&\ \,\vdots
\end{aligned}$$

where ϵ is a small parameter and $|i\rangle = \frac{1}{\sqrt{i!}}(a^\dagger)^i|0\rangle$. P_n can now be written as

$$P_n = \sum_{\alpha,\beta=1}^{n} |u_\alpha\rangle\,\langle u_\alpha|\,u_\beta\rangle^{-1}\,\langle u_\beta| = \sum_{i=0}^{n-1} |i\rangle\,\langle i| + \mathcal{O}(\epsilon)\,,$$

and hence, in the limit $\epsilon \to 0$, Φ_n describes n solitons from the 0 up to the $n-1$ harmonic oscillator level, all at the origin.

In the generic case of different z_α, P_n is unitarily equivalent to the projector onto the subspace spanned by the vectors $|i\rangle$ with $i < n$. To construct the unitary transformation explicitly, we diagonalize the matrix A (which is hermitean and positive semidefinite). Let $\vec{v}_{(\alpha)}$ and $a_{(\beta)}$ denote its (orthonormalized) eigenvectors and corresponding eigenvalues:

$$A_{\alpha\beta}\,v_{\beta(\gamma)} = a_{(\gamma)}\,v_{\alpha(\gamma)}\,, \qquad \vec{v}_{(\alpha)} \cdot \vec{v}_{(\beta)} = \delta_{\alpha\beta}\,. \tag{2.7}$$

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Defining

$$W_{\alpha\beta} = \frac{v_{\alpha(\beta)}}{\sqrt{a_{(\beta)}}} \tag{2.8}$$

and the Hilbert space vectors

$$|w_\alpha\rangle = \sum_{\beta=1}^{n} |z_\beta\rangle\, W_{\beta\alpha}\,, \tag{2.9}$$

we find

$$\langle w_\alpha | w_\beta\rangle = \delta_{\alpha\beta}\,, \qquad \text{and} \qquad \Phi_n = \lambda \sum_{\alpha=1}^{n} |w_\alpha\rangle\,\langle w_\alpha|\,.$$

For $n = 2$ this procedure — as expected — gives $\{|w_1\rangle, |w_2\rangle\} = \{|z_+\rangle, |z_-\rangle\}$.

Finally, we take any orthonormal basis in $\mathcal{H}$ whose first n vectors coincide with $|w_\alpha\rangle$ and denote it by $\{|w_j\rangle\}$. For

$$U = \sum_{j=1}^{\infty} |w_j\rangle\,\langle j-1| \tag{2.10}$$

we have

$$UU^\dagger = U^\dagger U = \mathbf{1}\,,$$

and

$$\Phi_n = \lambda U \left(\sum_{i=0}^{n-1} |i\rangle\,\langle i|\right) U^\dagger\,. \tag{2.11}$$

In a completely analogous manner, one may construct excited multisoliton solutions from states $|z_\alpha, n\rangle = (a^\dagger)^n\,|z_\alpha\rangle$ (the excited two-soliton case was worked out in detail in [16]). However, as shown in section 4, such solitons are all unstable for any non-zero θ, and hence we do not discuss them further.

2.2 Moduli spaces

The metric on the multi-soliton moduli space is Kähler for any n. Up to a constant normalization factor we have [16]:

$$\begin{aligned} g_{z_\alpha z_\beta} &= \frac{1}{\lambda^2}\,\mathrm{Tr}\left(\partial_{z_\alpha}\Phi_n \partial_{z_\beta}\Phi_n\right)\,,\\ g_{z_\alpha \bar z_\beta} &= \frac{1}{\lambda^2}\,\mathrm{Tr}\left(\partial_{z_\alpha}\Phi_n \partial_{\bar z_\beta}\Phi_n\right)\,,\\ g_{\bar z_\alpha \bar z_\beta} &= \frac{1}{\lambda^2}\,\mathrm{Tr}\left(\partial_{\bar z_\alpha}\Phi_n \partial_{\bar z_\beta}\Phi_n\right)\,. \end{aligned} \tag{2.12}$$

Straightforward calculation gives

$$g_{z_\alpha z_\beta} = g_{\bar z_\alpha \bar z_\beta} = 0\,,$$

while

$$g_{z_\alpha \bar{z}_\beta} = A^{-1}_{\alpha\beta}\left(A_{\beta\alpha} + \bar{z}_\beta A_{\beta\alpha} z_\alpha - \sum_{\gamma,\delta=1}^{n} A_{\beta\gamma} z_\gamma A^{-1}_{\gamma\delta} \bar{z}_\delta A_{\delta\alpha}\right) = \partial_{z_\alpha}\partial_{\bar{z}_\beta} K(\mathbf{z},\bar{\mathbf{z}}) \tag{2.13}$$

with

$$K(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\alpha=1}^{n} |z_\alpha|^2 + \log\det A\,. \tag{2.14}$$

2.3 The three-soliton case

As a simple example, we consider the three-soliton metric in detail; the matrix A becomes

$$\begin{pmatrix} 1 & e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2+\bar{z}_1 z_2} & e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_3|^2+\bar{z}_1 z_3} \\ \text{c.c.} & 1 & e^{-\frac{1}{2}|z_2|^2-\frac{1}{2}|z_3|^2+\bar{z}_2 z_3} \\ \text{c.c.} & \text{c.c.} & 1 \end{pmatrix}, \tag{2.15}$$

and gives the Kähler potential (using that $z_3 = -z_1 - z_2$)

$$\begin{aligned} K = \ln\Big(& e^{2|z_1|^2+2|z_2|^2+\bar{z}_1 z_2+\bar{z}_2 z_1} + e^{-|z_1|^2-|z_2|^2+\bar{z}_1 z_2-2\bar{z}_2 z_1} + e^{-|z_1|^2-|z_2|^2-2\bar{z}_1 z_2+\bar{z}_2 z_1} - \\ & - e^{-2|z_1|^2+|z_2|^2-\bar{z}_1 z_2-\bar{z}_2 z_1} - e^{|z_1|^2-2|z_2|^2-\bar{z}_1 z_2-\bar{z}_2 z_1} - \\ & - e^{|z_1|^2+|z_2|^2+2\bar{z}_1 z_2+2\bar{z}_2 z_1}\Big)\,. \end{aligned} \tag{2.16}$$

This has the two-soliton metric as a subspace: if we take one of the solitons (say z_2) far away and fix its position we can choose the coordinates to be

$$\begin{aligned} z_1 &= -\frac{z_2}{2} + \zeta\,, \\ z_3 &= -\frac{z_2}{2} - \zeta\,, \end{aligned} \tag{2.17}$$

where ζ is the relative distance between the soliton at z_1 and the soliton at z_3 and is taken to be much smaller than z_2. Inserting this in (2.16) we get

$$K \longrightarrow \frac{3}{2}|z_2|^2 + \ln\left(2\sinh(2\bar{\zeta}\zeta)\right). \tag{2.18}$$

Thus the geometry factorizes into two pieces, one, coordinatized by z_2, which is flat, and another, coordinatized by ζ, with precisely the Kähler potential of the 2 soliton moduli space, including the conical singularity when the solitons coincide.

We can also study what happens when all three solitons come together. We study the most symmetrical case where the solitons are at the same distance from the origin and separated by the angle $2\pi/3$. In that case we can choose the coordinates

$$\begin{aligned} z_1 &= \zeta\,, \\ z_2 &= \omega\zeta\,, \\ z_3 &= \omega^2\zeta\,, \\ \omega &= e^{2\pi i/3}\,. \end{aligned} \tag{2.19}$$

For small values of ζ we get the Kähler potential

$$K = \ln\left(|\zeta|^6 + \frac{|\zeta|^{12}}{120} + \cdots\right), \tag{2.20}$$

giving rise to a conical singularity of the type

$$ds^2 = r^4\left(dr^2 + r^2 d\theta^2\right). \tag{2.21}$$

This lead to a scattering angle of $2\pi/3$.

3. Finite θ: perturbation theory

We now consider the finite θ corrections to the soliton solutions.

At finite θ one has to include the derivative terms in the energy functional:

$$E = \frac{2\pi}{g^2}\,\mathrm{Tr}\left(\frac{1}{2}[a\,,\phi][\phi\,,a^\dagger\,] + \theta V(\phi)\right). \tag{3.1}$$

If we make the ansatz

$$\phi = \lambda\left(P + \frac{1}{\theta}B\right) \tag{3.2}$$

and use the conditions $V'(0) = V'(\lambda) = 0$, then (3.1) gives[1]

$$\begin{aligned} E = \frac{2\pi}{g^2}\,\mathrm{Tr}\Bigg[&\theta V(\lambda)P + \frac{\lambda^2}{2}[\,a\,,P\,][P, a^\dagger\,] + \\ &+ \frac{\lambda^2}{\theta}\Bigg(B\,[[P,a^\dagger\,],a\,]\frac{1}{2}V''(\lambda)PBPBP + \\ &\qquad + \frac{1}{2}V''(0)(1-P)B(1-P)B(1-P)\Bigg)\Bigg] + \mathcal{O}\left(\theta^{-2}\right). \end{aligned} \tag{3.3}$$

Extremizing with respect to the perturbation B, to leading order we find

$$V''(\lambda)PBP + V''(0)\,(1-P)\,B\,(1-P) = [\,a\,,[\,P, a^\dagger\,]] \equiv \Delta_P\,. \tag{3.4}$$

This implies that P has to satisfy the consistency condition

$$[P, \Delta_P\,] = 0\,. \tag{3.5}$$

Fortunately, this relation is fulfilled for all the soliton solutions we consider.

We can write the solution of (3.4) in the form

$$B = \frac{1}{V''(\lambda)}P\Delta_P P + \frac{1}{V''(0)}(1-P)\Delta_P(1-P) + PX(1-P) + (1-P)X^\dagger P\,, \tag{3.6}$$

where X is an arbitrary operator; it drops out of both (3.4) and — as a consequnece of (3.5) — the energy (3.3) to order θ^{-1}, and hence we choose $X = 0$ in what follows.

[1] No order θ^{-2} terms in ϕ contribute to the energy at order θ^{-1} because they are multiplied by $V'(\lambda)$ or $V'(0)$, which both vanish.

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A special class of projectors that satisfies (3.5) is given by solutions of the equation

$$(1-P)aP = 0\,. \tag{3.7}$$

It includes the projectors that give the n soliton solutions (2.4).

It is trivial to see that (3.7) implies (3.5), while the reverse implication does not hold — the projector $|n\rangle\langle n|$ with $n > 0$, satisfies (3.5) but not (3.7).

For operators P satisfying (3.7), the energy (3.3) can be rewritten in a simpler form

$$\begin{aligned} E = {} & \frac{2\pi}{g^2}\,\mathrm{Tr}\left(\theta V(\lambda)P + \frac{\lambda^2}{2}P\right) - \\ & -\frac{\pi\lambda^2}{\theta g^2}\left(\frac{1}{V''(\lambda)} + \frac{1}{V''(0)}\right)\mathrm{Tr}\left[2(Pa^\dagger a)^2 - 2P(a^\dagger)^2a^2 + P\right]. \end{aligned} \tag{3.8}$$

3.1 Explicit solutions

For one-soliton states, our perturbative expansion reproduces the results in [1, appendix A] to the appropriate order in θ^{-1}.

For $P = |z+\rangle\langle z+| + |z-\rangle\langle z-|$ (2.1), the simplifying condition (3.7) is satisfied. One may calculate the first order correction B to Φ from (3.6):

$$B = \frac{1}{V''(\lambda)}P\Delta_P P + \frac{1}{V''(0)}(1-P)\Delta_P(1-P)\,, \tag{3.9}$$

by substituting $\Delta_P = [\,a\,,[\,P, a^\dagger\,]]$ as above, but to find the corrections to the energy, it is easier to use the expression (3.9), simplify using the cyclicity of the trace, and the condition (3.7), and only then substitute the explicit form of P.

At order θ the energy is a constant:

$$E_0 = \frac{2\pi}{g^2}2\theta V(\lambda)\,. \tag{3.10}$$

One might expect that at the lowest non-trivial order the energy could depend on the relative position of the solitons z, but this dependence cancels:

$$E_1 = \frac{4\pi}{g^2}\lambda^2\,. \tag{3.11}$$

(For the unstable excited states of [16], $E_1 = \frac{2\pi}{g^2}\lambda^2(4n+2)$.) The z-independence at the lowest non-trivial order implies there is a range of energies for which the moduli space is an accurate description even at finite θ. At next order we find

$$E_2 = -\frac{\pi\lambda^2}{g^2\theta}\frac{V''(\lambda)+V''(0)}{V''(\lambda)V''(0)}\left\{2 + \left(\frac{2z\bar{z}}{\sinh(2z\bar{z})}\right)^2\right\}. \tag{3.12}$$

The z-dependent part of this produces an attractive force between the solitons. However, it is very short-range, vanishing as $e^{-4|z|^2}$. For small $|z|$, the potential between the solitons goes smoothly to a finite constant. (One may consider solitons such that the false vacuum is a maximum rather than a minimum: $V''(\lambda) < 0$. Then the potential between the solitons could be repulsive or vanish; however, in this case the solitons are unstable.)

3.2 Corrections to the moduli space metric

Using the perturbative scheme developed above, we can compute the leading corrections to the level zero n soliton metric (2.13). To order θ^{-1} we have:

$$g_{\alpha\bar{\beta}} = \frac{1}{\lambda^2}\,\mathrm{Tr}\left(\partial_\alpha \Phi \bar{\partial}_\beta \Phi\right) = g^{(0)}_{\alpha\bar{\beta}} + \frac{2}{\theta} g^{(1)}_{\alpha\bar{\beta}} + \mathcal{O}(\theta^{-2})\,, \tag{3.13}$$

where

$$\begin{aligned} g^{(1)}_{\alpha\bar{\beta}} &= \frac{1}{2}\left(\mathrm{Tr}\left(\partial_\alpha P_n \bar{\partial}_\beta B_n\right) + \mathrm{Tr}\left(\partial_\alpha B_n \bar{\partial}_\beta P_n\right)\right), \\ B_n &= \left(\frac{1}{V''(\lambda)} P_n + \frac{1}{V''(0)}(1 - P_n)\right)[a\,,[\,P_n\,,a^\dagger\,]\,]\,, \end{aligned} \tag{3.14}$$

and $\partial_\alpha \equiv \partial_{z^\alpha}$ and $\bar{\partial}_\beta \equiv \partial_{\bar{z}^\beta}$. For

$$P_n = \sum_{\alpha,\beta=1}^{n} |z_\alpha\rangle\, A^{-1}_{\alpha\beta}\, \langle z_\beta| \tag{3.15}$$

we find:

$$\begin{aligned} g^{(1)}_{\alpha\bar{\beta}} = {} & \frac{1}{V''(\lambda)}\left(z_\alpha A^{-1}_{\alpha\beta}\bar{z}_\beta - A^{-1}_{\alpha\beta} - \sum_{\lambda,\kappa=1}^{n} A^{-1}_{\alpha\lambda}\bar{z}_\lambda A_{\lambda\kappa} z_\kappa A^{-1}_{\kappa\beta}\right) \times \\ & \times \left(\bar{z}_\beta A_{\beta\alpha} z_\alpha + A_{\beta\alpha} - \sum_{\gamma,\lambda=1}^{n} A_{\beta\gamma} z_\gamma A^{-1}_{\gamma\lambda}\bar{z}_\lambda A_{\lambda\alpha}\right) - \\ & - \frac{1}{V''(0)} A^{-1}_{\alpha\beta}\left(A_{\beta\alpha} + 2\bar{z}_\beta A_{\beta\alpha} z_\alpha + \sum_{\lambda,\kappa,\gamma,\lambda=1}^{n} A_{\beta\lambda} z_\lambda A^{-1}_{\lambda\kappa}\bar{z}_\kappa A_{\kappa\gamma} z_\gamma A^{-1}_{\gamma\lambda}\bar{z}_\lambda A_{\lambda\alpha} - \right. \\ & \left. - \sum_{\sigma,\kappa=1}^{n}\left(2 + z_\alpha\bar{z}_\kappa + z_\lambda\bar{z}_\beta - z_\alpha\bar{z}_\beta\right) A_{\beta\lambda} z_\lambda A^{-1}_{\lambda\kappa}\bar{z}_\kappa A_{\kappa\alpha}\right). \end{aligned} \tag{3.16}$$

To find the metric on the two-soliton relative moduli space, we set $z^1 = z$, $z^2 = -z$, and hence

$$g_{z\bar{z}} \propto \left(g_{1\bar{1}} - g_{1\bar{2}}\right). \tag{3.17}$$

Explicitly,

$$A = \begin{pmatrix} 1 & \mathrm{e}^{-2|z|^2} \\ \mathrm{e}^{-2|z|^2} & 1 \end{pmatrix}, \tag{3.18}$$

and (3.16) gives

$$g^{(1)}_{z\bar z} = \frac{\coth r^2}{V''(\lambda)}\left[\left(\frac{r^2}{\sinh r^2}\right)^2 - 1\right] + \frac{\coth r^2}{V''(0)}\left[\frac{4r^2}{\sinh 2r^2} - \left(\frac{r^2}{\sinh r^2}\right)^2 - 1\right], \tag{3.19}$$

where $r^2 = 2z\bar z$. Note that for $V''(\lambda) = V''(0)$, this correction to f takes exactly the same functional form as the original f.

4. Stability analysis

In [1] it was shown that there exists a path in field space interpolating between field configurations corresponding to the operators $|n\rangle\langle n|$ and $|0\rangle\langle 0|$ and along which the gradient energy decreases monotonically. This path is given by $|\alpha\rangle\langle\alpha|$, $0 \le \alpha \le \pi/2$, where

$$|\alpha\rangle = \cos\alpha\,|n\rangle + \sin\alpha\,|0\rangle\,. \tag{4.1}$$

Hence for finite θ the state $|n\rangle\langle n|$ decays to the state $|0\rangle\langle 0|$.

However, for finite θ the state $|n\rangle\langle n|$ differs from the stationary point of the full static energy functional by terms of order θ^{-1}, and the energy of the true solution is smaller then the energy corresponding to the $|n\rangle\langle n|$ state by terms of the same order. Due to this energy difference the true level n-soliton cannot decay along the $|\alpha\rangle\langle\alpha|$ path. Its instability can be, however, still demonstrated with the help of the perturbative methods discussed in the section 3.

We define

$$P_\alpha \equiv |\alpha\rangle\langle\alpha|\,, \qquad \Delta_\alpha = [a\,,[P_\alpha\,,a^\dagger\,]\,]\,. \tag{4.2}$$

P_α satisfies the consistency condition

$$[\,P_\alpha\,,\Delta_\alpha\,] = 0 \tag{4.3}$$

only for α an integer multiple of $\pi/2$, but we can still consider a path in field space given by:

$$\Phi(\alpha) = \lambda\left[P_\alpha + \frac{1}{\theta}\left(\frac{1}{V''(\lambda)}P_\alpha\Delta_\alpha P_\alpha + \frac{1}{V''(0)}(1-P_\alpha)\Delta_\alpha(1-P_\alpha)\right)\right]. \tag{4.4}$$

$\Phi(0)$ and $\Phi(\pi/2)$ are (up to the terms of order θ^{-2}) the true level n- and level 0-solitons, and the energy of the $\Phi(\alpha)$ configuration is given by (for $n > 1$)

$$\begin{aligned} E(\alpha) = \frac{2\pi\theta}{g^2}V(\lambda) + \frac{\pi\lambda^2}{g^2}\Big[1 + 2n\cos^2\alpha - \frac{1}{\theta V''(\lambda)}\left(1+2n\cos^2\alpha\right)^2 - \\ - \frac{1}{\theta V''(0)}\left(1 + 2n\cos^2\alpha + 2n^2\cos^4\alpha\right)\Big] + \mathcal{O}\left(\theta^{-2}\right). \end{aligned}$$

When $\theta V'' \gg n$, the function $E(\alpha)$ decreases monotonically for $0 \leq \alpha \leq \pi/2$, and hence the $|n\rangle\langle n|$ soliton is unstable for $n > 1$. For $n = 1$ the resulting formula for $E(\alpha)$ is slightly different, but the conclusion is the same.

The analysis above can be extended to the case of the solutions corresponding at $\theta = \infty$ to the projection operators

$$P_I = \sum_{k \in I} |k\rangle\langle k|\,, \qquad I \subset \mathbb{N}. \tag{4.5}$$

If there is a "gap" in the set I, i.e. for some $m < n$ we have $m \notin I$ and $n \in I$, then using the projectors

$$P_{m,n}(\alpha) = (\cos\alpha\ |n\rangle + \sin\alpha\ |m\rangle)\left(\langle n|\cos\alpha + \langle m|\sin\alpha\right), \tag{4.6}$$

we can construct a "decay path" as in (4.4). This shows that potentially stable, radially symmetric, level n-solitons $\phi_{(n)}$ must approach

$$\phi^{\infty}_{(n)} = \lambda \sum_{k=0}^{n} |k\rangle\langle k| \tag{4.7}$$

as $\theta \to \infty$. We now consider the stability of such states.

For ψ of the (general) form

$$\psi\left(\{f_k\}, U\right) = \lambda \sum_{k=0}^{\infty} f_k\ U^\dagger |k\rangle\langle k| U\,,$$

where U is some unitary operator, the energy functional (3.1) can be written as

$$\begin{aligned} E[\psi] = \frac{2\pi}{g^2}\sum_{k=0}^{\infty}\left[\theta V(\lambda f_k) + \frac{1}{2}\lambda^2 f_k^2\right] + \frac{2\pi\lambda^2}{g^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} l f_k^2\, |U_{k,l}|^2 - \\ - \frac{2\pi\lambda^2}{g^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f_k f_l \left|\sum_{p=0}^{\infty}\sqrt{p+1}\,U_{k,p}U^*_{l,p+1}\right|^2, \end{aligned} \tag{4.8}$$

with $U_{p,k} \equiv \langle p|U|k\rangle$.

The radially symmetric states $\phi_{(n)}$ have the form

$$\phi_{(n)} = \lambda \sum_{k=0}^{\infty} c_k\, |k\rangle\langle k|\,, \tag{4.9}$$

where, for large enough θ, the perturbative analysis gives

$$c_k = \begin{cases} 1 + \mathcal{O}\left(\theta^{-2}\right) & k < n\,, \\ 1 - \frac{2(n+1)}{\theta V''(\lambda)} + \mathcal{O}\left(\theta^{-2}\right) & k = n\,, \\ \frac{2(n+1)}{\theta V''(0)} + \mathcal{O}\left(\theta^{-2}\right) & k = n+1\,, \\ \mathcal{O}\left(\theta^{-2}\right) & k > n+1\,. \end{cases} \tag{4.10}$$

To check stability of this solution we consider

$$\phi_{(n)} + \delta\phi = \lambda \sum_{k=0}^{\infty} \left(c_k + \varepsilon \delta c_k\right) U^\dagger \left|k\right\rangle \left\langle k\right| U \,, \tag{4.11}$$

where δc_k are arbitrary real parameters and $U \equiv \mathrm{e}^{i\varepsilon T}$ with arbitrary hermitean T. Using (4.8) we find

$$\begin{aligned}
\frac{g^2}{2\pi} E[\phi_{(n)} + \delta\phi] = \theta \sum_{k=0}^{\infty} V(c_k) + \lambda^2 \sum_{k=0}^{\infty} \left(\left(k + \frac{1}{2}\right) c_k^2 - (k+1)\, c_k c_{k+1} \right) + \\
+ \frac{1}{2} \varepsilon^2 \lambda^2 \sum_{k=0}^{\infty} \left(\theta V''(\lambda c_k) \left(\delta c_k\right)^2 + (k+1) \left(\delta c_{k+1} - \delta c_k\right)^2 \right) + \\
+ \varepsilon^2 \lambda^2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \Big\{ (l-k) c_k^2 \left|T_{k,l}\right|^2 - \\
- \; c_k c_l \left| \sqrt{l} T_{k,l-1} - \sqrt{k+1} T_{k+1,l} \right|^2 + \\
+ \; c_k c_{k+1} \Big[(k+1) \left(\left|T_{k,l}\right|^2 + \left|T_{k+1,l}\right|^2 \right) - \\
- \sqrt{(k+1)(l+1)} \left(T_{k,l} T^*_{k+1,l+1} + \text{c.c.} \right) \Big] \Big\} + \\
+ \mathcal{O}\left(\varepsilon^3\right) .
\end{aligned} \tag{4.12}$$

The second line of this equation shows that as long as $V''(\lambda c_k) > 0$ for all k, $\phi_{(n)}$ is stable against "radial" perturbations $c_k \to c_k + \varepsilon \delta c_k$.

To check stability against unitary rotations we use the perturbative form of $\phi_{(n)}$ (4.9) and (4.10). We denote by $\mathcal{E}$ the T-dependent part of (4.12). Terms of order θ^0 in $\mathcal{E}$ can be written in the manifestly positive semi-definite form:

$$\mathcal{E}|_{\theta^0} = \sum_{k=0}^{n} \sum_{l=n+2}^{\infty} \left| \sqrt{l} T_{k,l} - \sqrt{k} T_{k-1,l-1} \right|^2 . \tag{4.13}$$

In general, these dominate any terms that are lower order in θ; however, (4.13) has zero-modes, so we need to consider terms of order θ^{-1}:

$$\begin{aligned}
\mathcal{E}|_{\theta^{-1}} = \frac{2(n+1)}{\theta V''(\lambda)} \left[\sum_{k=0}^{n} \left| \sqrt{n+1} T_{k,n+1} - \sqrt{k} T_{k-1,n} \right|^2 - \sum_{l=0}^{\infty} (l-n) |T_{l,n}|^2 \right] + \\
+ \frac{2(n+1)}{\theta V''(0)} \left[\sum_{l=n+1}^{\infty} \left| \sqrt{n+1} T_{n,l} - \sqrt{l} T_{n+1,l+1} \right|^2 - \sum_{l=0}^{\infty} (l-n-1) |T_{l,n+1}|^2 \right] - \\
- \frac{2(n+1)}{\theta V''(\lambda)} \sum_{l=n+2}^{\infty} \left| \sqrt{l} T_{n,l} - \sqrt{n} T_{n-1,l-1} \right|^2 - \\
- \frac{2(n+1)}{\theta V''(0)} \sum_{k=0}^{n} \left| \sqrt{n+2} T_{k,n+2} - \sqrt{k} T_{k-1,n+1} \right|^2 .
\end{aligned} \tag{4.14}$$

Some of these are positive as they stand, and some are obviously dominated by the $\mathcal{O}(\theta^0)$ terms, e.g. the negative sum proportional to $\frac{1}{V''(\lambda)}$ together with the terms in (4.13) with $k = n$ can be written in the form

$$\left(1 - \frac{2(n+1)}{\theta V''(\lambda)}\right) \sum_{l=n+2}^{n} \left|\sqrt{l}T_{n,l} - \sqrt{n}T_{n-1,l-1}\right|^2 ,$$

which is positive for $\frac{2(n+1)}{\theta V''(\lambda)} < 1$. Using the identity

$$\left|\sqrt{n+1}T_{n,n+1} - \sqrt{n}T_{n-1,n}\right|^2 + |T_{n-1,n}|^2 - |T_{n,n+1}|^2 = \left|\sqrt{n}T_{n,n+1} - \sqrt{n+1}T_{n-1,n}\right|^2$$

we can rewrite the remaining, proportional to $\frac{1}{V''(\lambda)}$ terms in (4.14) as

$$\frac{2(n+1)}{\theta V''(\lambda)} \left\{ \left|\sqrt{n}T_{n,n+1} - \sqrt{n+1}T_{n-1,n}\right|^2 + \sum_{k=1}^{n-1} \left|\sqrt{n+1}T_{k,n+1} - \sqrt{k}T_{k-1,n}\right|^2 + \right.$$
$$\left. + (n+1)|T_{0,n+1}|^2 + \sum_{k=0}^{n-2}(n-k)|T_{k,n}|^2 - \sum_{l=n+2}^{\infty} (l-n)|T_{n,l}|^2 \right\} . \qquad (4.15)$$

These terms are potentially dangerous only along the zero mode direction of $\mathcal{E}|_{\theta^0}$ with some non-vanishing $T_{n,l}, l \geq n+2$. Such modes obey the condition

$$\sum_{k=0}^{n} \sum_{l=n+2}^{\infty} \left|\sqrt{l}T_{k,l} - \sqrt{k}T_{k-1,l-1}\right|^2 = 0 , \qquad (4.16)$$

which gives $|T_{0,l}| = 0$ for $l \geq n+2$ and the recursion relation

$$T_{k,l} = \sqrt{\frac{k}{l}}T_{k-1,l-1} ; \qquad (4.17)$$

this implies

$$T_{n,l} = 0 \qquad \text{for} \qquad l > 2n+1 .$$

For $l = 2n+1$ (4.17) gives

$$(n+1)\left|T_{n,2n+1}\right|^2 = \frac{((n+1)!)^2}{(2n+1)!}\left|T_{0,n+1}\right|^2 .$$

while for $l = n+p$ with $2 \leq p \leq n$ we get

$$\left|T_{n,n+p}\right|^2 = \frac{(n!)^2}{(n-p)!(n+p)!}\left|T_{n-p,n}\right|^2$$

and the last three terms in (4.15) give

$$(n+1)|T_{0,n+1}|^2 + \sum_{k=0}^{n-2}(n-k)|T_{k,n}|^2 - \sum_{l=n+2}^{\infty}(l-n)|T_{n,l}|^2 =$$

$$= (n+1)\left(|T_{0,n+1}|^2 - |T_{n,2n+1}|^2\right) + \sum_{p=2}^{n} p\left(|T_{n-p,n}|^2 - |T_{n,n+p}|^2\right)$$

$$= \left(n+1-\frac{((n+1)!)^2}{(2n+1)!}\right)|T_{0,n+1}|^2 + \sum_{p=2}^{n} p\left(1-\frac{(n!)^2}{(n-p)!(n+p)!}\right)|T_{n-p,n}|^2 > 0\,.$$

Similar analysis applies to the terms proportional to $\frac{1}{V''(0)}$.

The remaining zero modes of (4.12) (apart from the exact, translational zero mode which — for $\theta \to \infty$ — is just $T_{n,n+1}$) are given by $T_{k,l}$ with $k > n+1$ or with $l < n$ (thanks to the hermiticity of T we can always choose k to be smaller than l). The first set of these zero modes (with $k > n$) is irrelevant — they correspond to the unitary transformations that do not change the state to order θ^{-1}.

The other set of zero modes (with $k < l < n$) corresponds to rotations that act on the non-zero c_k's and do change the soliton. To check if they can destabilize the soliton we would have to extend the perturbative analysis to higher orders in θ^{-1}. Fortunately, they are absent for the most interesting case, that of two solitons (i.e. for $n = 1$).

5. The geodesic equation at finite θ

We now study classical scattering in the presence of a generated potential U.

5.1 Integrating the classical equations of motion

For concreteness and simplicity we consider the two-soliton case (3.12):

$$U(r) = -\left(\frac{1}{2\theta}\right)\frac{V''(\lambda)+V''(0)}{V''(\lambda)V''(0)}\left(\frac{r^2}{\sinh(r^2)}\right)^2. \qquad (5.1)$$

The effective action for the two-soliton system is

$$\frac{2\pi\lambda^2}{g^2}\int dt\left(\frac{\theta}{2}\left(g_{rr}\dot{r}^2 + g_{\vartheta\vartheta}\dot{\vartheta}^2\right) - U(r)\right), \qquad (5.2)$$

where the metric g is given by $ds^2 = f(r)(dr^2 + r^2 d\vartheta^2)$ and f is as in [16]

$$f(r) = \coth(r^2) - \frac{r^2}{\sinh^2(r^2)}\,. \qquad (5.3)$$

In principle, we should consider the corrections (3.19) to the moduli space metric; as discussed below, for large θ, these can be ignored. In contrast, the potential U is important even though it is also suppressed at large θ.

Varying the action leads to the equations of motion

$$\begin{aligned} \ddot{r} + \Gamma^r_{rr}\dot{r}^2 + \Gamma^r_{\vartheta\vartheta}\dot{\vartheta}^2 + \frac{g^{rr}}{\theta}\frac{dU}{dr} &= 0\,, \\ \ddot{\vartheta} + 2\Gamma^\vartheta_{r\vartheta}\dot{r}\dot{\vartheta} &= 0\,. \end{aligned} \tag{5.4}$$

The second equation is the same as for the case without U and the solution is [16]

$$\dot{\vartheta} = \frac{l}{r^2 f(r)}\,, \tag{5.5}$$

where l is an integration constant corresponding to the angular momentum. The first equation can then be written as

$$\ddot{r} + \frac{1}{2}\frac{d}{dr}\ln(f)\dot{r}^2 = \frac{1}{2f}\frac{d}{dr}(r^2 f)\left(\frac{l}{r^2 f}\right)^2 - \frac{1}{\theta f}\frac{dU}{dr}\,. \tag{5.6}$$

An integrating factor for this equation is $\dot{r}f$ from which one finds the solution

$$f\dot{r}^2 = 2\frac{E-U}{\theta} - \frac{l^2}{r^2 f}\,, \tag{5.7}$$

where E is an integration constant with the interpretation of the total energy of the system. In the same way as in the case without a potential [16], this leads to scattering trajectories found from the integral

$$\vartheta(r) = -\int_\infty^r \frac{ds}{s\sqrt{(s/b)^2 f(s)(1-\frac{U}{E}) - 1}} \tag{5.8}$$

here $b^2 = \frac{l^2\theta}{2E}$ is the impact parameter as in [16] (the θ dependence arises because we have rescaled the coordinates). The finite θ correction to the geodesic scattering picture can therefore be found by using a corrected function

$$\tilde{f}(r) = f(r)\left(1 - \frac{U(r)}{E}\right). \tag{5.9}$$

Since U is attractive (negative) there are no extra divergencies in the effective $\tilde{f}$ as compared to f. If U had been repulsive, but of the same functional form,[2] it would have made the effective $\tilde{f}$ more repulsive.

We can make some estimates of the validity of our approximations by restoring the dimensions of the coordinates: $r \to r/\sqrt{\theta}$. Since $E = \frac{\lambda^2}{g^2}(\frac{f}{2}v^2 + U)$, which corresponds to a particle of effective mass $f\lambda^2/g^2$ moving in a potential U, we can

[2]This could happen for a potential V where the false vacuum corresponds to a maximum; however, such solitons are unstable.

find a range of velocities where the moduli space approximation should be good. For the correction to the classical result in (5.8) to be small we need $U/E \ll 1$ leading to

$$v^2 \gg \frac{1}{2\theta}\frac{V''(\lambda)+V''(0)}{V''(\lambda)V''(0)}\,. \tag{5.10}$$

However, for the adiabatic approximation to be valid, the momentum transfer must remain sufficiently small so that fluctuations out of the moduli space are suppressed. In our case there are several possiblities since potentials for different fluctuations appear at different orders in perturbation theory. Even if we do not have the exact potentials, we can estimate their strength from the general behavior of perturbation theory. Most fluctuations have potentials already at $\theta = \infty$. They are not excited as long as

$$v^2 \ll \frac{\theta V(\lambda)}{\lambda^2}\,. \tag{5.11}$$

Other fluctuations get a potential only at first order in perturbation theory. They are not excited as long as

$$v^2 \ll 1\,; \tag{5.12}$$

this simply means that the motion remains non relativistic. For the two-soliton case, we have checked that the fluctuations with lower energies correspond to motions of the solitons, which we do not want to restrict. Higher soliton scattering requires a higher-order analysis.

5.2 Trajectories

It is interesting to investigate some explicit cases for the scattering trajectories of the previous section. We have prepared movie clips in MPEG format.[3] The first movie shows the behavior for large values of the impact parameter. The solitons just pass each other with no scattering taking place. In the second movie the right angle scattering for small impact parameter b is shown.[4] Notice that this qualitative behavior is true irrespective of the value of the total energy E since it only depends on the value of the function $\tilde{f}$ at large or small r. However, in the presence of the attractive potential $U(r)$ and for small enough energy ($U_0/E > 3.86$) we find new qualitative behavior shown in the third movie. We get a metastable orbit where the solitons circle around each other for some time before they scatter to infinity. These results are summarized in the following picture where the exit angle is plotted as a function of the impact parameter in the case where $U_0/E = 5$.

[3]If the reader's viewer does not support hypertex, the three movies can be found at `http://www.physto.se/~unge/traj1.mpg`, `http://www.physto.se/~unge/traj2.mpg`, and `http://www.physto.se/~unge/traj3.mpg`.

[4]The second movie has time slowed down by a factor of 1000; to keep the file a manageable size, a smaller spatial region is shown.

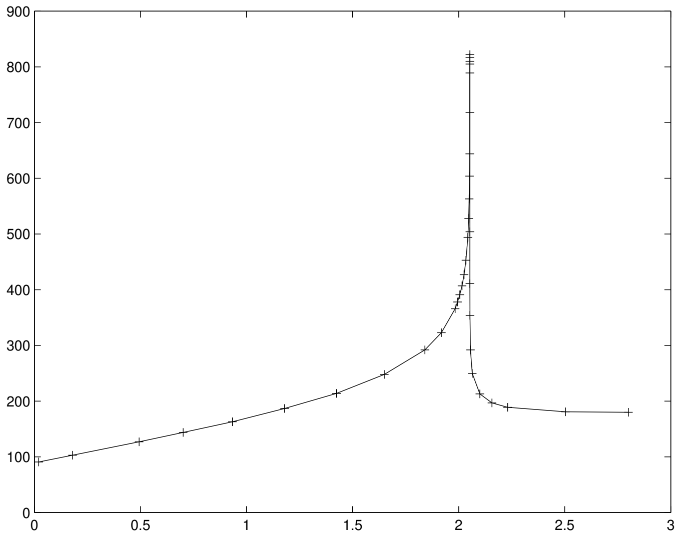

Figure 1: The exit angle in degrees as a function of the impact parameter for $U_0/E = 5$.

One may also ask what happens when we include the corrections to the moduli space metric (3.19). When $V''(\lambda) = V''(0)$, f is rescaled by $(1 + \frac{2\lambda}{\theta})$, which can be absorbed by a redefinition of b. However, when $V''(\lambda)$ and $V''(0)$ are not equal we find two different behaviors: if $V''(\lambda) > V''(0)$, the trapped orbit effect above is suppressed, that is, it appears for smaller E, or larger U_0/E, than before. On the other hand, if $V''(0) > V''(\lambda)$ we get an enhancement of the effect. In fact, trapped orbits appear even for $U_0 = 0$ if $V''(\lambda)$ is small enough!

6. Quantization

In this section we discuss the quantization of the effective hamiltonian that describes the motion of solitons. We focus on the two-soliton case. The Schrödinger equation for this problem can be written as[5]

$$\left(-\nabla^2 + U\right)\psi = E\psi\,, \tag{6.1}$$

where

$$\nabla^2 = \frac{1}{fr}\partial_r\left(r\partial_r\right) + \frac{1}{fr^2}\partial_\vartheta^2\,, \tag{6.2}$$

and $f(r)$ is the metric (5.3). In the absence of a potential, this hamiltonian operator is positive;[6] consequently, it cannot have any bound states. For any attractive potential, no matter how small, an s-wave bound state appears. Thus the potential that we

[5] There is really a factor of $\frac{1}{2\sqrt{\theta}}$ in front of the ∇^2 operator but we can soak it up in a redefinition of U and E so that they become dimensionless.

[6] We thank Matthew Headrick of reminding us of this fact and pointing out that by ignoring it in a previous version of this paper, we had made a serious error.

found perturbatively (3.12) induces such a bound state for any finite value of θ. The potential is rotationally symmetric and hence for $\psi = \chi(r)e^{il\vartheta}$ the equation reduces to:

$$\left(-\frac{1}{fr}\partial_r\left(r\partial_r\right) + \frac{l^2}{fr^2} + U\right)\chi = E\chi\,. \tag{6.3}$$

We may estimate the energy of the bound state as follows: for sufficiently small r, $f \approx \frac{2}{3}r^2$ and $U \approx U_0 < 0$. This admits a solution with energy E of the form of a Bessel function $J_0(kr^2)$ for $k^2 = (U_0 - E)/6$. For large r, $f \approx 1$ and $U \approx 0$; this admits a solution of the form $A\, K_0(\sqrt{-E}\, r)$, where A is some normalization constant and K_0 is a Bessel function of the second kind. Matching these at some intermediate value r_m where $U(r_m) \to E$ gives

$$\frac{2kr_m J_1(kr_m^2)}{J_0(kr_m^2)} = \frac{\sqrt{-E}\, K_1(\sqrt{-E}\, r_m)}{K_0(\sqrt{-E}\, r_m)}\,; \tag{6.4}$$

for small enough $|U_0|$, $|E|$, we find

$$E \approx -\frac{1}{r_m}\, e^{-2/(U_0 r_m^4)}\,. \tag{6.5}$$

Of course, the details of $f(r)$ and $U(r)$ correct the solution and the energy, but they cannot change the qualitative behavior.

Because of this s-wave bound state, for sufficiently small soliton energy the cross section becomes very large; this ruins the moduli space picture. The conclusion is therefore the same as in the previous section: for the moduli space picture to be a good approximation we need energies in an intermediate range, not big enough to ruin the adiabatic approximation but not so small as to see the bound state.

The question of how the classical metastable states found in the previous section appear in the quantum treatment is left to future work.

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Note added. After completing our work, [24] appeared. This work has significant overlap with ours. The authors observe that the solitons naturally obey Bose-Einstein statistics, and that imposing this on the solitons replaces the moduli space with its quotient by the symmetric group; as pointed out in [16] (where, however, we did *not* impose Bose-Einstein statistics), this quotient smoothes the singularities discussed in section 2. In the quantum analysis of section 6, Bose-Einstein statistics require us to consider only even angular momentum l. Finally, we want to thank the referee for making us aware of [25], where non-commutative solitons are considered from a somewhat different perspective.

References

[1] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[2] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[3] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[4] D. Bak, *Exact multi-vortex solutions in noncommutative abelian-Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204].

[5] D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078].

[6] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *J. High Energy Phys.* **04** (2001) 001 [hep-th/0009142].

[7] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[8] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as noncommutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

[9] S.S. Gubser and S.L. Sondhi, *Phase structure of non-commutative scalar field theories*, hep-th/0006119.

[10] V. Pasquier, *Skyrmions in the quantum hall effect and noncommutative solitons*, *Phys. Lett.* **B 490** (2000) 258 [hep-th/0007176].

[11] J.H. Brodie, L. Susskind and N. Toumbas, *How Bob Laughlin tamed the giant graviton from Taub-Nut space*, *J. High Energy Phys.* **02** (2001) 003 [hep-th/0010105].

[12] S.S. Gubser and M. Rangamani, *D-brane dynamics and the quantum hall effect*, *J. High Energy Phys.* **05** (2001) 041 [hep-th/0012155].

[13] L. Susskind, *The quantum hall fluid and non-commutative Chern Simons theory*, `hep-th/0101029`.

[14] A.P. Polychronakos, *Quantum hall states as matrix chern-simons theory*, *J. High Energy Phys.* **04** (2001) 011 [`hep-th/0103013`].

[15] S. Hellerman and M.V. Raamsdonk, *Quantum hall physics equals noncommutative field theory*, `hep-th/0103179`.

[16] U. Lindstrom, M. Roček and R. von Unge, *Non-commutative soliton scattering*, *J. High Energy Phys.* **12** (2000) 004 [`hep-th/0008108`].

[17] A. Solovyov, *On noncommutative solitons*, *Mod. Phys. Lett.* **A 15** (2000) 2205 [`hep-th/0008199`].

[18] C.-G. Zhou, *Noncommutative scalar solitons at finite theta*, `hep-th/0007255`.

[19] B. Durhuus, T. Jonsson and R. Nest, *Noncommutative scalar solitons: existence and nonexistence*, *Phys. Lett.* **B 500** (2001) 320 [`hep-th/0011139`].

[20] M.G. Jackson, *The stability of noncommutative scalar solitons*, `hep-th/0103217`.

[21] M. Li, *Quantum corrections to noncommutative solitons*, `hep-th/0011170`.

[22] Talk by S.-J. Rey at strings 2001.

[23] Talk by R. Gopakumar at strings 2001.

[24] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons*, `hep-th/0103256`.

[25] E.J. Martinec and G. Moore, *Noncommutative solitons on orbifolds*, `hep-th/0101199`.